PATTERN-EQUIVARIANT COHOMOLOGY WITH INTEGER COEFFICIENTS

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Abstract. We relate Kellendonk and Putnam's pattern-equivariant (PE) cohomology to the inverse-limit structure of a tiling space. This gives an version of PE cohomology with integer coefficients, or with values in any Abelian group. It also provides an easy proof of Kellendonk and Putnam's original theorem relating PE cohomology to the Čech cohomology of the tiling space. The inverse-limit structure also allows for the construction of a new non-Abelian invariant, the PE representation variety.

1. Background

Kellendonk and Putnam's theory of pattern-equivariant cohomology [K] (henceforth abbreviated PE cohomology) gives a wonderfully intuitive tool for understanding the Čech cohomology of a tiling space. Via Ruelle-Sullivan currents [KP], it is also a powerful tool for understanding the natural action of the translation group on such a space. The only limitation is that PE cohomology is defined using differential forms, and so necessarily has real coefficients (or coefficients in a real vector space). All of the additional information contained in the integer-valued Čech cohomology, such as torsion and divisibility, is lost.

In this paper, we present a version of PE cohomology with integer coefficients, or indeed with values in any Abelian group. This cohomology is isomorphic to the Čech cohomology of a tiling space with integer (or group) coefficients. The proof of this fact is quite simple, and is easily modified to give a simpler proof of Kellendonk and Putnam's original theorem, namely that their cohomology is isomorphic to the Čech cohomology of the tiling space with *real* coefficients. Along the way, we also prove a de Rham theorem for branched manifolds: if the branched manifold is obtained by gluing polygons along edges, then the de Rham cohomology of the branched manifold is isomorphic to the Čech cohomology with real coefficients.

Finally, using the concept of pattern-equivariance, and following an idea of Klaus Schmidt, we define a new topological invariant of tiling spaces, called the PE representation variety of the tiling space. This generalizes, to non-Abelian groups, the first cohomology of the tiling space with group coefficients. It is related to, but different from, invariants defined by Geller and Propp [GP] and by Schmidt [Sch].

Kellendonk and Putnam originally defined PE cohomology for point patterns. However, tilings and point patterns are closely related. From any tiling, we can get a point

1991 *Mathematics Subject Classification.* Primary: 37B50, Secondary: 52C23, 54F65, 37C85.

pattern by considering the vertices; from any point pattern, we can get a tiling by considering Voronoi cells. This paper is phrased entirely in terms of tilings, and the edges and faces of the tilings will be essential ingredients in the constructions.

We consider tilings of $\mathbb{R}^d$ that meet three conditions: (1) there are only a finite number of tile types, up to translation, (2) the tiles are polygons (in $d = 2$ dimensions, or polyhedra in higher dimensions), and (3) the tiles meet full-face to full-face. The second and third conditions may at first appear severe, but any tiling that has finite local complexity (FLC) with respect to translation is mutually locally derivable (MLD) to a tiling that meets these conditions. A tiling with FLC is associated with a Delone point pattern with FLC, which in turn is associated with a derived Voronoi tiling that meets the three conditions.

The translation group $\mathbb{R}^d$ acts naturally on a tiling. If $x \in \mathbb{R}^d$ and T is a tiling, then $T - x$ is the tiling T translated backwards by x. That is, a neigborhood of the origin in $T-x$ looks like a neighborhood of x in T. Two tilings are considered ϵ-close if they agree, up to a rigid translation by ϵ or less, on a ball of radius $1/\epsilon$ around the origin. Given a tiling T, the *tiling space* of T, or the *continuous hull* of T, denoted Ω_T, is the closure (in the space of all tilings) of the translational orbit of T. Another characterization of Ω_T is the following: a tiling T' is in Ω_T if and only if every patch of T' is found (up to translation) somewhere in T. Since T has finite local complexity, Ω_T is compact. If T is *repetitive*, meaning that every pattern in T repeats with bounded gaps, then Ω_T is a minimal dynamical system, i.e., every translational orbit in Ω_T is dense.

Let T be a tiling of $\mathbb{R}^d$. Two points $x, y \in \mathbb{R}^d$ are *T-equivalent to radius r* if $T - x$ and $T - y$ agree exactly on a ball of radius r around the origin. A function of $\mathbb{R}^d$ is *PE with radius r* with respect to T if there exists an $r > 0$ such that $f(x) = f(y)$ whenever x and y are T-equivalent to radius r. A differential k-form on $\mathbb{R}^d$ is PE if it can be written as a finite sum $\sum_I f_I(x) dx^I$, where each $f_I(x)$ is a PE function and $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ is a constant k-form. (In [KP] a distinction is made between functions that meet the above conditions, which are called *strongly* PE, and *weakly* PE functions that are uniform limits of strongly PE functions. In this paper we will only consider strongly PE functions.)

Let $\Lambda^k_P(T)$ denote the set of PE k-forms. It is easy to see that the exterior derivative of a PE form is PE, and $d^2 = 0$, as usual, so we have a differential complex:

$$0 \longrightarrow \Lambda^0_P(T) \xrightarrow{d_0} \Lambda^1_P(T) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \Lambda^d_P(T) \xrightarrow{d_d} 0. \tag{1}$$

The pattern-equivariant cohomology of T is the cohomology of this complex:

$$H^k_P(T) = (\ker d_k)/(\mathrm{Im}(d_{k-1})). \tag{2}$$

Kellendonk and Putnam proved:

Theorem 1. *$H^*_P(T)$ is isomorphic to $\check{H}^*(\Omega_T, \mathbb{R})$, the Čech cohomology with real coefficients of the tiling space of T.*

In section 2 we review the structure of the tiling space Ω_T as an inverse limit of approximants Γ_k, and give a new proof of Kellendonk and Putnam's theorem. We will see that a PE form of $\mathbb{R}^d$ is just the pullback of a form on Γ_k, and that the PE cohomology of T is just the direct limit of the de Rham cohomology of the approximants. Thanks to

a de Rham theorem for branched manifolds (proved in the appendix), this is the same as the direct limit of the Čech cohomology of the approximants with real coefficients, and hence equal to the Čech cohomology of the inverse limit.

In section 3 we use the CW structure of $\mathbb{R}^d$ induced by the tiling T to define PE cellular cochains with values in an arbitrary Abelian group. Repeating the argument of section 2, these are the pullbacks of cellular cochains on approximants Γ_k, so the PE cohomology of T is isomorphic to the direct limit of the cellular cohomology of Γ_k, hence to the direct limit of the Čech cohomology of Γ_k, hence to the Čech cohomology of Ω_T.

In section 4 we define a new topological invariant, the PE representation variety of a tiling with gauge group G. This is defined intrinsically in terms of PE connections and PE gauge transformations, but is seen to equal the direct limit $\lim_{\rightarrow} Hom(\pi_1(\Gamma_n), G)/G$. For each approximant, the representation variety is derived from the fundamental group, but in the limit the fundamental group disappears, while the representation variety remains.

2. Inverse limits and real cohomology

There are several constructions of tiling spaces as inverse limits [AP, ORS, BBG, BG, Sa]. For our purposes, the most useful is Gähler's construction (unpublished), building on the earlier work of Anderson and Putnam [AP]. We sketch the construction below; see [Sa] for further details and generalization.

A point in the n-th approximant Γ_n is a description of a tile containing the origin, its nearest neighbors (sometimes called the "first corona"), its second nearest neighbors (the "second corona") and so on out to the n-th nearest neighbors. The map $\Gamma_n \to \Gamma_{n-1}$ simply forgets the n-th corona. A point in the inverse limit is then a consistent prescription for constructing a tiling out to infinity. In other words, it defines a tiling.

What remains is to construct Γ_n out of geometric pieces. We consider two tiles t_1, t_2 in T to be equivalent if a patch of T, containing t_1 and its first n coronas, is identical, up to translation, to a similar patch around t_2. Since T has finite local complexity, there are only finitely many equivalence classes, each of which is called an n-collared tile. (Note that an n-collared tile is no bigger than an ordinary tile, but its label contains information on how the first n coronas are laid out around it.)

For each n-collared tile t_i, we consider how such a tile can be placed around the origin. This is tantamount to picking a point in t_i to call the origin, so the set of ways to place t_i is just a copy of t_i itself.

A patch of a tiling in which the origin is on the boundary of two or more tiles is described by points on the boundary of two or more n-collared tiles, and these points must be identified. The branched manifold Γ_n is the disjoint union of the n-collared tiles t_i, modulo this identification. Each of the points being identified carries complete information about the placement of all the tiles that meet the origin, together with their first $n-1$ coronas.

Γ_n is called the *Anderson-Putnam complex* with n-collared tiles. This is a branched manifold, with branches where multiple tile boundaries are identified. At the branches, there is a well-defined tangent space, with a global trivialization given by the action of the

translation group, so it makes sense to speak of smooth functions and smooth differential forms.

The tiling T induces a natural map $\pi_n : \mathbb{R}^d \to \Gamma_n$. For each $x \in \mathbb{R}^d$, $\pi_n(x)$ is the point in Γ_n that describes a neighborhood of the origin in $T - x$, or equivalently a neighborhood of x in T. Let L be the diameter of the largest tile. If x and y are T-equivalent to radius R, and if $R > (n+1)L$, then $\pi_n(x) = \pi_n(y)$. Likewise, if $\pi_n(x) = \pi_n(y)$, then there is a radius r (which grows uniformly with n) such that x and y are T-equivalent to radius r. Thus a function f of $\mathbb{R}^d$ is PE if and only if there exists an n such that $f(x) = f(y)$ whenever $\pi_n(x) = \pi_n(y)$. That is, there exists a function on Γ_n whose pullback is f, in which case there also exist functions on all $\Gamma_{n'}$ with $n' > n$ whose pullback is f. The same argument applies to differential forms. That is:

Theorem 2. $\Lambda^k_P(T) = \cup_n \pi_n^*(\Lambda^k(\Gamma_n))$,

where $\Lambda^k(\Gamma_n)$ denotes the set of smooth k-forms on Γ_n.

Our last ingredient for proving Theorem (1) is the de Rham theorem for branched manifolds:

Theorem 3. *If X is a branched manifold obtained by gluing tiles along their common boundaries, then the de Rham cohomology of X is naturally isomorphic to the Čech cohomology of X with real coefficients.*

Proof: See appendix.

Proof of Theorem (1): Every PE cohomology class is represented by a closed PE form ω, which is the pullback of a closed form $\omega^{(n)}$ on Γ_n, which defines a de Rham cohomology class $[\omega^{(n)}]_{dR}$ on Γ_n, which defines a Čech cohomology class on Γ_n. This defines a class in the direct limit of the Čech cohomologies of the approximants, which is the Čech cohomology of the inverse limit space Ω_T. In other words, we have a map $\phi : H^*_P(T) \to \check{H}^*(\Omega_T)$. We must show that ϕ is well-defined, 1–1 and onto.

Being well-defined means being independent of the choices made, namely of n and of the representative ω. If we pick a different approximant $\Gamma_{n'}$, with $n' > n$, then there are commuting maps:

$$\begin{tikzcd} & \Omega_T \arrow[dl, "\pi_{n'}"'] \arrow[dr, "\pi_n"] & \\ \Gamma_{n'} \arrow[rr, "\rho_{nn'}"'] & & \Gamma_n \end{tikzcd}$$

The form $\omega^{(n')}$ is the pullback by the forgetful map $\rho_{nn'}$ of the form $\omega^{(n)}$. By the naturality of the isomorphism between de Rham and Čech cohomology, the Čech class defined by $\omega^{(n')}$ is the pullback of the Čech class defined by $\omega^{(n)}$, and hence defines the same element of the direct limit of Čech cohomologies.

Next suppose that ω' and ω define the same PE class of T. Then $\omega' = \omega + d\nu$, where ν is a PE form. Then there exists an n such that ω, ν and ω' are all pullbacks of forms on Γ_n. But then $\omega'^{(n)} = \omega^{(n)} + d\nu^{(n)}$, so ω and ω' define the same de Rham cohomology

class on Γ_n, hence the same Čech class, and hence the same class in the direct limit of cohomologies. This shows that ϕ is well-defined.

To see that ϕ is 1–1, suppose that $\phi([\omega]) = 0$. Then there exists a finite n such that the Čech class defined by $\omega^{(n)}$ is zero, so the de Rham class defined by $\omega^{(n)}$ is zero, so there is a form ν on Γ_n with $d\nu = \omega^{(n)}$, so $[\omega] = [d\pi_n^*(\nu)] = 0$ in PE cohomology.

Finally, every class γ in the direct limit of Čech cohomologies is represented by a Čech class in some Γ_n, which in turn is represented by a closed form on Γ_n. Pulling this back to $\mathbb{R}^d$, we get a closed PE form, hence a PE class whose image under ϕ is γ. ∎

3. Integer-valued PE cohomology

We next construct PE cohomology with integer coefficients. Although we speak of integers, the same arguments apply, with no loss of generality, to coefficients in any Abelian group.

A tiling is more than a point pattern. A tiling T gives a CW decomposition of $\mathbb{R}^d$ into 0-cells (vertices), 1-cells (edges), 2-cells (faces), and so on. We may therefore speak of cellular chains with the obvious boundary maps, and we can dualize to get cellular cochains with integer (or group) coefficients. Likewise, each approximant Γ_n is constructed as a union of tiles, glued along boundaries. This, too, is a CW complex, and the map $\pi_n : \mathbb{R}^d \to \Gamma_n$ is cellular.

For $k > 0$, we say that two k-cells c_1 and c_2 are *T-equivalent to radius r* if there exist interior points $x \in c_1$ and $y \in c_2$ such that x and y are T-equivalent to radius r. (T-equivalence for 0-cells has already been defined.) We say a cochain α on $\mathbb{R}^d$ is PE if there exists an $r > 0$ such that $\alpha(c_1) = \alpha(c_2)$ whenever c_1 and c_2 are equivalent to radius r. The coboundary δ of a PE cochain is PE (albeit with a slightly larger radius). Denoting the PE k-cochains on $\mathbb{R}^d$ by $C^k_P(T)$, we have a complex

$$0 \longrightarrow C^0_P(T) \xrightarrow{\delta_0} C^1_P(T) \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{n-1}} C^d_P(T) \xrightarrow{\delta_d} 0. \tag{3}$$

and define the integer-valued PE cohomology of T to be the cohomology of this complex.

Theorem 4. *The integer-valued PE cohomology of T is isomorphic to the Čech cohomology of Ω_T.*

Proof: As before, we see that a cochain on $\mathbb{R}^d$ is PE if and only if it is the pullback of a cochain on Γ_n for some n (and hence any sufficiently large n). Every PE cohomology class is represented by a closed PE cochain γ, which is the pullback of a closed cochain $\gamma^{(n)}$ on Γ_n, which defines a cellular cohomology class $[\gamma^{(n)}]$ on Γ_n, which defines a Čech cohomology class on Γ_n, which defines a class in the direct limit of the Čech cohomologies of the approximants, which is the Čech cohomology of the inverse limit space Ω_T.

Showing that this compound map from $H^*_P(T)$ to $\check{\mathrm{H}}^*(\Omega_T)$ is well-defined and an isomorphism proceeds exactly as in the proof of Theorem (1), with the word "form" replaced by "cochain", "exterior derivative" replaced by "coboundary", and "de Rham theorem" replaced by "the natural isomorphism between cellular and Čech cohomology", which holds for all CW complexes. ∎

4. PE representation varieties

Gauge theory provides a wealth of topological invariants of manifolds. Given a manifold M and a connected Lie Group G, we can consider the moduli space of flat connections on the (trivial) principal bundle $M \times G$, modulo gauge transformation. At first glance this moduli space might appear to be only an invariant of the smooth structure of M, but in fact it is a topological invariant. Flat connections are described by their holonomy, i.e., by a homomorphism $\pi_1(M) \to G$, and gauge equivalent connections have maps that are related by conjugation by an element of G. In other words, the moduli space is isomorphic to the *representation variety* $Hom(\pi_1(M), G)/G$, where the quotient is by conjugation. (The representation variety is well-defined even when G is not connected, and is a non-Abelian generalization of the first cohomology of M. When G is Abelian, $Hom(\pi_1(M), G)/G = Hom(\pi_1(M), G) = Hom(H_1(M), G) = H^1(M, G)$.)

Using pattern equivariance, we can play the same game for tilings. Given a tiling T and a connected Lie group G, we consider flat PE connections on the trivial principal bundle $\mathbb{R}^d \times G$, modulo PE gauge transformations. We call this the *PE representation variety* of the tiling T. As with PE cohomology, the PE representation variety is defined abstractly, without reference to inverse limits, but is most easily understood in terms of inverse limits and Gähler's construction.

If A is a PE connection 1-form with zero curvature, then A is the pullback of a connection 1-form on Γ_n with zero curvature, and so defines an element of the representation variety $Hom(\pi_1(\Gamma_n), G)/G$. Fundamental groups do not behave well under inverse limits, but their duals do. For $n' > n$, we have a forgetful map $\Gamma_{n'} \to \Gamma_n$, which induces a map $\pi_1(\Gamma_{n'}) \to \pi_1(\Gamma_n)$, which induces a map $Hom(\pi_1(\Gamma_n), G) \to Hom(\pi_1(\Gamma_{n'}), G)$. This commutes with the G action, and induces a map of representation varieties: $Hom(\pi_1(\Gamma_n), G)/G \to Hom(\pi_1(\Gamma_{n'}), G)/G$. The PE representation variety of T is the direct limit $\lim_{\rightarrow} Hom(\pi_1(\Gamma_n), G)/G$.

Notice that for each approximant Γ_n, the representation variety contained no additional information to that in the fundamental group of Γ_n. However, this changes in the infinite limit. The fundamental group of Ω_T is trivial, but the PE representation variety of T can be nontrivial. Indeed, when G is Abelian, the PE representation variety of T is the first Čech cohomology of Ω_T with coefficients in G.

It is natural to ask whether there is a limiting object, dual to the PE representation variety, that plays a role analagous to the fundamental group. For $\mathbb{Z}^d$ actions on subshifts of finite type, something akin to this was considered by Geller and Propp [GP]. Their tiling group is somewhat smaller than $\pi_1(\Gamma_n)$, as they only consider paths that correspond to closed loops in $\mathbb{R}^d$. That is, they consider the kernel of the map $\pi_1(\Gamma_1) \to \mathbb{R}^d$, $\gamma \to \int_\gamma d\vec{x}$. They then consider the *projective fundamental group* of a tiling to be the direct limit of the tiling groups of the approximants. The cohomology class of Schmidt's *fundamental cocycle* [Sch] is a non-Abelian structure dual to Geller and Propp's tiling group. As such, it should be closely related to the PE representation variety, but this relation is not yet understood.

5. Acknowledgements

Many thanks to Jean Bellissard, Johannes Kellendonk, Ian Putnam, Bob Williams and especially Klaus Schmidt for helpful discussions. In particular, the idea for the PE representation variety grew directly out of discussions with Schmidt. This work is partially supported by the National Science Foundation.

6. Appendix: A de Rham theorem for branched manifolds

Let X be a branched manifold obtained by gluing Euclidean polygons (or polyhedra) along their common faces, as in the Anderson-Putnam complex. Note that the branch set is locally star-shaped with respect to every point on the branch set. That is, for each point p, there is a ball U_p around p such that the branch set, intersected with U_p, is a union of (Euclidean) line segments with endpoint p. If p is a vertex, we take U_p to be a ball of radius less half than the distance from p to the nearest other vertex. (This ensures that the neighborhoods corresponding to different vertices do not intersect, a condition that will prove useful.) If p is on an edge, we take the radius of U_p to be less than half the distance to the nearest other edge, and also less than the distance to the nearest vertex. If p is on a face, we take the radius to be less than half the distance to the nearest other face, and also less than the distance to the nearest edge or vertex.

Theorem 5 (Poincaré Lemma)**.** *Let $U = U_{p_1} \cap U_{p_2} \cap \cdots \cap U_{p_n}$ be a non-empty intersection of a finite number of balls U_{p_i} constructed as above. Then U has the de Rham cohomology of a point. That is, $H^0_{dR}(U) = \mathbb{R}$, and $H^k_{dR}(U) = 0$ for $k > 0$.*

Proof: If U does not touch the branch set of X, then U is a convex subset of a single polygon, and the usual proof of the Poincaré Lemma applies. We therefore assume that U hits the branch set of X. But by the construction of U_{p_i}, this means that all the points p_i must lie on the same k-cell C, and that the branch set, intersected with U, is $U \cap C$. Since the branch set of each U_{p_i} is a convex ball in C, the branch set of U is a convex set in C, and in particular is star-shaped. Let p_0 be any point in $U \cap C$.

The proof of the Poincaré Lemma found in [Spi], page 94, then carries over almost word for word. We define a homotopy operator I that takes k-forms to $k-1$ forms such that, for any k-form ω, $d(I\omega) + I(d\omega) = \omega$. If ω is closed, then $\omega = dI\omega$ is exact. The form $I\omega$ is computed by integrating ω out along a straight line from p_0. Since the straight line is either entirely in the branch set or entirely outside the branch set, the value of $I(\omega)$ at a point does not depend on which coordinate disk is used to do the calculation. Specifically,

$$(I\omega)_{i_1,\ldots,i_{k-1}}(x) = \sum_j \int_0^1 t^{k-1}(x-p_0)_j \omega_{j,i_1,i_2,\ldots,i_{k-1}}(p_0 + t(x-p_0))dt. \quad \blacksquare \tag{4}$$

Now take an open cover $\mathcal{U}$ of X, with each open set of the form U_{p_i}.

Theorem 6. *The de Rham cohomology of X is isomorphic to the Cech cohomology of $\mathcal{U}$ with real coefficients.*

Proof: We set up the Čech -de Rham double complex as in Chapter II of [BT]. The rows are all exact, by the partition-of-unity argument found in ([BT], Proposition 8.5). This implies that the cohomology of the double complex is isomorphic to the de Rham cohomology of X. Likewise, by the Poincaré Lemma, the columns are all exact, and the cohomology of the double complex is isomorphic to the Čech cohomology of the cover with real coefficients. ∎

Proof of Theorem 3: The Cech cohomology of X is, by definition, the direct limit of the Cech cohomology of the open covers of X. However, every open cover has a refinement of the form $\mathcal{U}$, since each set U_{p_i} is made by taking an arbitrarily small ball around the point p_i. In computing the Cech cohomology of X, it is thus sufficient to consider only covers of the form $\mathcal{U}$. For each of these, theorem 6 gives an isomorphism to de Rham. So each term in the direct limit is the same, and each map is an isomorphism. ∎

Remark: The proof given here can easily be generalized to broader classes of branched manifolds. The only necessity is for the branch set to be sufficiently well-behaved that one can prove a Poincare Lemma. An arbitrary branched surface may not have a well-behaved branch set, but Williams [Wi] showed how to construct inverse limit spaces using only "nice" branched manifolds, for which a theorem similar to 3 should be expected to hold.

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